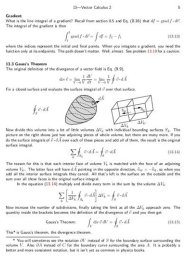

13—Vector Calculus 2 5

Gradient

What is the line integral of a gradient? Recall from section 5.5 and Eq. (3.30) that $df = \text{grad}\, f \cdot d\vec{r}$. The integral of the gradient is then

$$\int_C \text{grad}\, f \cdot d\vec{r} = \int_C df = f_2 - f_1 \tag{13.13}$$

where the indices represent the initial and final points. When you integrate a gradient, you need the function only at its endpoints. The path doesn't matter. Well, almost. See problem 13.10 for a caution.

13.3 Gauss's Theorem

The original definition of the divergence of a vector field is Eq. (3.9),

$$\text{div}\, \vec{v} = \lim_{\Delta\tau \to 0} \frac{\Delta\tau}{\Delta\tau} = \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} \oint_S \vec{v} \cdot d\vec{a}.$$

For a closed surface and evaluate the surface integral of $\vec{v}$ over that surface.

Now divide this volume into a lot of little volumes ΔV_i with individual bounding surfaces S_{ip}. The picture on the right shows just two adjoining pieces of which volume, but there are many more. If you do the surface integrals of $\vec{v} \cdot d\vec{a}$ over each of these pieces and add all of them, the result is the original surface integral

$$\sum_i \oint_{S_{ip}} \vec{v} \cdot d\vec{a} = \oint_S \vec{v} \cdot d\vec{a}. \tag{13.14}$$

The reason for this is that each interior face of volume V_{ip} is matched with the face of an adjoining volume V_{iq}. The latter face will have $d\vec{a}$ pointing in the opposite direction, $S_{iq} = -S_{ip}$, so when you add up the two surface integrals, the result is that it left in the surface on the outside and the sum over all these faces is the original surface integral.

In the equation (13.14) multiply and divide every term in the sum by the volume ΔV_i,

$$\sum_i \left[\frac{1}{\Delta V_i} \oint_{S_{ip}} \vec{v} \cdot d\vec{a} \right] \Delta V_i = \oint_S \vec{v} \cdot d\vec{a}$$

Now increase the number of subdivisions, finally taking the limit as all the ΔV_i approach zero. The quantity inside the brackets becomes the definition of the divergence of $\vec{v}$ and you then get

$$\text{Gauss's Theorem:} \qquad \int_V \text{div}\, \vec{v}\, dV = \oint_S \vec{v} \cdot d\vec{a} \tag{13.15}$$

This* is Gauss's theorem, the divergence theorem.

* You will sometimes see the notation $\oint V$ instead of $\oint$ for the boundary surface surrounding the volume V. Also $\oint\!\!\oint$ instead of $\oint$ for the boundary curve surrounding the area A. It is probably a better and more convenient notation, but it isn't yet as common in physics books.

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